Two-Group Neutron Transport Theory for Plane and Spherical Geometry *

K. O. THIELHEIM and W. BLÖCKER

Institut für Reine und Angewandte Kernphysik, University of Kiel, West Germany

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Two-Group neutron transport theory is applied to critical problems in plane and spherical geometry. The neutron flux and the density transform for plane and spherical geometry respectively are expanded into singular eigenfunctions of the transport equation. With aid of the theory of singular integral equations the problem is reduced to one Fredholm integral equation for the expansion coefficients. The critical equations are presented as additional conditions.

1. Introduction

Exact solutions of multigroup transport theory are of interest for comparison with results of approximative methods. For this purpose a two-group transport theory of critical slabs and spheres adequate for numerical computations is presented in the present paper.

The two-group neutron transport theory in plane geometry has been discussed by ZELAZNY and KUSZELL [1], who generalized the monoenergetic normal mode expansion technique to two groups of energy. These authors have reduced the problem of infinite critical slabs without reflector to a system of coupled Fredholm integral equations with additional conditions of solubility. Unfortunately this procedure is not convenient for numerical computations. The general scheme of two-group transport theory has been developed by SIEWERT and SHIEH [2], who succeeded in giving explicitly Green's function for infinite homogeneous media. The monoenergetic theory of a bare critical sphere has been discussed by MITSIS [3].

In Sect. 2 of the present paper, eigenvalues and eigenfunctions of the two-group transport equation in plane geometry for isotropic scattering will be presented. The integral transformation of the Boltzmann equation for spherical geometry generating a transport equation, which is formally identical with the one in plane geometry, will be given in Section 3. In Sect. 4, the theory of MUSKHELISHVILI [4] will be applied to the expansion of the neutron flux and the density transform into eigenfunctions for plane and spherical geometry, respectively. A Fredholm integral equation for the expansion coefficients and a critical equation will be given.

2. Transport Equation in Plane Geometry and Eigenfunctions

The one-dimensional transport equation for two energy groups of neutrons in a homogeneous medium with isotropic scattering is

$$\mu \frac{\partial}{\partial x} \boldsymbol{\Psi}(x,\mu) + \sum \boldsymbol{\Psi}(x,\mu) = \boldsymbol{C} \int_{-1}^{1} \boldsymbol{\Psi}(x,\mu')\,\mathrm{d}\mu' \quad (1)$$

with the two-component differential neutron flux

$$\boldsymbol{\Psi}(x,\mu) = \begin{pmatrix} \psi_1(x,\mu) \\ \psi_2(x,\mu) \end{pmatrix},$$

the total cross-section matrix $\sum = \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma > 1$,

and the transfer matrix $\boldsymbol{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$.

From the separation ansatz introduced by CASE [5]

$$\boldsymbol{\Psi}(x,\mu) = e^{-x/\eta}\, \boldsymbol{F}(\eta,\mu) \quad (2)$$

we are led to the following sets of singular eigenfunctions. In the continuous range of eigenvalues $\eta \in [-1, 1]$ we find two linearly independent sets of eigenfunctions:

$$\boldsymbol{F}_1^{(1)}(\eta,\mu) = \begin{pmatrix} -c_{12}\,\delta(\eta\sigma-\mu) \\ \dfrac{c\,\eta}{\eta-\mu} + \delta(\eta-\mu)\left(c_{11} + \eta\, c\, L(\eta\sigma)\right) \end{pmatrix}, \quad (3)$$

$$\boldsymbol{F}_2^{(1)}(\eta,\mu) = \begin{pmatrix} \dfrac{c\,\eta}{\eta\sigma-\mu} + \delta(\eta\sigma-\mu)\left(c_{22} + \eta\, c\, L(\eta\sigma)\right) \\ -c_{21}\,\delta(\eta-\mu) \end{pmatrix}. \quad (4)$$

In the range $\eta \in [-1, -1/\sigma]$ or $[1/\sigma, 1]$ we obtain one set of eigenfunctions

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$$\boldsymbol{F}^{(2)}(\eta,\mu)=\begin{pmatrix}\dfrac{c_{12}\,\eta}{\eta\,\sigma-\mu}\\[2mm] \dfrac{\eta(c_{22}+\eta\,c\,L(\eta\,\sigma))}{\eta-\mu}+\delta(\eta-\mu)\,P\,\Omega(\eta)\end{pmatrix} \tag{5}$$

with $L(\eta)=\ln\left|\dfrac{\eta-1}{\eta+1}\right|$, $c=\det\boldsymbol{C}$.

$P\,\Omega(\eta)$ denotes the principal value of the function

$$\Omega(z)=1+z\,c_{22}\int_{-1}^{1}\frac{d\eta}{\eta-z}+z\,c_{11}\int_{-1}^{1}\frac{d\eta}{\eta-z\,\sigma} \tag{6}$$
$$+z^2\,c\left(\int_{1}^{-1}\frac{d\eta}{\eta-z}\right)\left(\int_{-1}^{1}\frac{d\eta}{\eta-z\,\sigma}\right).$$

The set of discrete eigenvalues $\pm\eta_s\notin[-1,1]$ is given by the roots of the dispersion equation

$$\Omega(\pm\eta_s)=0\,, \tag{7}$$

which have been studied by SIEWERT and SHIEH [2]. There exist either one or two pairs $\pm\eta_s$, which may be real or imaginary. The corresponding eigenfunctions are

$$\boldsymbol{F}(\pm\eta_s,\mu)=\begin{pmatrix}\dfrac{c_{12}\,\eta_s}{\eta_s\,\sigma\mp\mu}\\[2mm] \dfrac{c_{22}+\eta_s\,c\,L(\eta_s\,\sigma)}{\eta_s\,\sigma\mp\mu}\,\eta_s\end{pmatrix}. \tag{8}$$

The completeness of the preceding sets of eigenfunctions for the range $\mu\in[0,1]$ corresponding to the range $\eta\in[0,1]$ has been proved by BARAN [6] and independently by METCALF and ZWEIFEL [7].

3. Transport Equation in Spherical Geometry

Since a complete set of eigenfunctions is not yet known for the differential transport equation in spherical geometry, we start from the integral two-group transport equation for a critical sphere of radius R:

$$\varrho_i(r)=\frac{1}{2\pi}\iiint_{r'<R}\frac{\exp\{-|\boldsymbol{r}-\boldsymbol{r}'|\,\sigma_i\}}{|\boldsymbol{r}-\boldsymbol{r}'|^2}\sum_j c_{ij}\,\varrho_j(r')\,d\boldsymbol{r}', \tag{9}$$

which can be transformed to

$$r\,\varrho_i(r)=\int_{-R}^{R}E_1(\sigma_i\,|\,r-r'\,|)\sum_j c_{ij}\,\varrho_j(r')\,dr' \tag{10}$$

with $\qquad \varrho_i(-r)=\varrho_i(r)\,.$

With aid of the well-known representation of the exponential integral

$$E_1(x)=\int_0^1(e^{-x/\mu}/\mu)\,d\mu$$

we are led to the following integral transform of the neutron density $\varrho_i(r)$:

$$\left.\begin{aligned}\psi_i(r,\mu)&=\frac{1}{\mu}\int_{-R}^{r}\exp\left\{-\frac{\sigma_i(r-t)}{\mu}\right\}\sum_j c_{ij}\,\varrho_j(t)\,t\,dt\\ \psi_i(r,-\mu)&=\frac{1}{\mu}\int_{r}^{R}\exp\left\{-\frac{\sigma_i(t-r)}{\mu}\right\}\sum_j c_{ij}\,\varrho_j(t)\,t\,dt\end{aligned}\right\}\quad\text{for}\quad\mu>0\,, \tag{11}$$

analogously to MITSIS [3]. The inverse transformation is

$$r\,\boldsymbol{\rho}(r)=\int_{-1}^{1}\boldsymbol{\Psi}(r,\mu)\,d\mu\,. \tag{12}$$

By differentiation of (11) with respect to r we obtain

$$\mu\frac{\partial}{\partial r}\boldsymbol{\Psi}(r,\mu)+\Sigma\,\boldsymbol{\Psi}(r,\mu)=\boldsymbol{C}\int_{-1}^{1}\psi(r,\mu')\,d\mu', \tag{13}$$

which is formally identical with the transport equation (1) in plane geometry. From the integral transform of the neutron density (11),

$$\boldsymbol{\Psi}(r,\mu)=-\boldsymbol{\Psi}(-r,-\mu) \tag{14}$$

is obtained.

4. Solution of Transport Equations for Critical Slabs and Spheres

Boundary conditions for critical slabs and spheres are

$$\boldsymbol{\Psi}(x,\mu)=\pm\boldsymbol{\Psi}(-x,-\mu), \tag{15}$$

$$\boldsymbol{\Psi}(-R,\mu)=0\qquad\text{for}\quad\mu>0, \tag{16}$$

respectively. The upper sign holds for plane geometry, the lower one for spherical geometry. Condition (15) leads to the expansion

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$$\begin{aligned}\boldsymbol{\Psi}(x,\mu) = A_0[&e^{-x/\eta_0}\boldsymbol{F}(\eta_0,\mu) \pm e^{x/\eta_0}\boldsymbol{F}(-\eta_0,\mu) + A_1(e^{-x/\eta_1}\boldsymbol{F}(\eta_1,\mu)\\ &\pm e^{x/\eta_1}\boldsymbol{F}(-\eta_1,\mu)) + P\int_0^{1/\sigma}\alpha(\eta)(e^{-x/\eta}\boldsymbol{F}_1^{(1)}(\eta,\mu) \pm e^{x/\eta}\boldsymbol{F}_1^{(1)}(-\eta,\mu))\,\mathrm{d}\eta\\ &+P\int_1^{1/\sigma}\beta(\eta)(e^{-x/\eta}\boldsymbol{F}_2^{(1)}(\eta,\mu) \pm e^{x/\eta}\boldsymbol{F}_2^{(1)}(-\eta,\mu))\,\mathrm{d}\eta\\ &+P\int_1^{1/\sigma}\varepsilon(\eta)(e^{-x/\eta}\boldsymbol{F}^{(2)}(\eta,\mu) \pm e^{x/\eta}\boldsymbol{F}^{(2)}(-\eta,\mu))\,\mathrm{d}\eta]\,.\end{aligned} \tag{17}$$

The contribution

$$A_1(e^{-x/\eta_1}\boldsymbol{F}(\eta_1,\mu) \pm e^{x/\eta_1}\boldsymbol{F}(-\eta_1,\mu))$$

vanishes, if only one pair of discrete eigenvalues, $\pm\eta_0$, exists. Inserting condition (16) into the expansion (17) results in the following system of coupled singular integral equations:

$$\begin{aligned}\boldsymbol{g}(\mu) \equiv\; &\mp e^{-R/\eta_0}\boldsymbol{F}(-\eta_0,\mu) - e^{R/\eta_0}\boldsymbol{F}(-\eta_0,\mu)\\ &-A_1(\pm e^{-R/\eta_1}\boldsymbol{F}(-\eta_1,\mu) + e^{R/\eta_1}\boldsymbol{F}(-\eta_0,\mu))\\ =\;&P\int_0^{1/\sigma}\alpha(\eta)(e^{R/\eta}\boldsymbol{F}_1^{(1)}(\eta,\mu) \pm e^{-R/\eta}\boldsymbol{F}_1^{(1)}(-\eta,\mu))\,\mathrm{d}\eta\\ &+P\int_0^{1/\sigma}\beta(\eta)(e^{R/\eta}\boldsymbol{F}_2^{(1)}(\eta,\mu) \pm e^{-R/\eta}\boldsymbol{F}_2^{(1)}(-\eta,\mu))\,\mathrm{d}\eta\\ &+P\int_{1/\sigma}^{1}\varepsilon(\eta)(e^{R/\eta}\boldsymbol{F}^{(2)}(\eta,\mu) \pm e^{-R/\eta}\boldsymbol{F}^{(2)}(-\eta,\mu))\,\mathrm{d}\eta \qquad \text{for} \quad \mu>0\,.\end{aligned} \tag{18}$$

Introducing the explicit forms of the eigenfunctions (3), (4), (5) we obtain the two components of (18)

$$\begin{aligned}g_1(\mu) = &-\frac{c_{12}}{\sigma}A(\mu/\sigma) + c\,P\int_0^{1/\sigma}\frac{\eta\,B(\eta)}{\eta\,\sigma-\mu}\mathrm{d}\eta\\ &+\frac{f_2(\mu/\sigma)}{\sigma}B(\mu/\sigma) \pm c\int_0^{1/\sigma}\frac{e^{-2R/\eta}B(\eta)\,\eta}{\eta\,\sigma+\mu}\mathrm{d}\eta\\ &+c_{12}\int_{1/\sigma}^{1}\frac{\eta\,D(\eta)}{\eta\,\sigma-\mu}\mathrm{d}\eta \pm c_{12}\int_{1/\sigma}^{1}\frac{e^{-2R/\eta}D(\eta)\,\eta}{\eta\,\sigma+\mu}\mathrm{d}\eta\,,\end{aligned} \tag{19}$$

and

$$\begin{aligned}g_2(\mu) = &\,c\,P\int_0^{1/\sigma}\frac{\eta\,A(\eta)}{\eta-\mu}\mathrm{d}\eta + \Theta(1/\sigma-\mu)\,A(\mu)\,f_1(\mu)\\ &\pm\int_0^{1/\sigma}\frac{e^{-2R/\eta}A(\eta)\,\eta}{\eta+\mu}\mathrm{d}\eta - \Theta(1/\sigma-\mu)\,c_{21}\,B(\mu)\\ &+P\int_{1/\sigma}^{1}\frac{\eta\,f_2(\eta)\,D(\eta)}{\eta-\mu}\mathrm{d}\eta\\ &+\Theta(\mu-1/\sigma)\,D(\mu)\,P\,\Omega(\mu)\\ &\pm\int_{1/\sigma}^{1}\frac{e^{-2R/\eta}f_2(\eta)\,\eta\,D(\eta)}{\eta+\mu}\mathrm{d}\eta\end{aligned} \tag{20}$$

with the abbreviations

$$\begin{aligned}&A(\eta) \equiv \alpha(\eta)\,e^{R/\eta}\,, \quad B(\eta) \equiv \beta(\eta)\,e^{R/\eta}\,, \quad D(\eta) \equiv \varepsilon(\eta)\,e^{R/\eta}\,,\\ &f_1(\eta) \equiv c_{11} + \eta\,c\,L(\eta)\,, \qquad f_2(\eta) \equiv c_{22} + \eta\,c\,L(\eta\,\sigma)\,,\end{aligned} \qquad \Theta(x) = \begin{cases}0 & \text{for} \quad x<0\,,\\ 1 & \text{for} \quad x>0\,.\end{cases} \tag{21}$$

After solving equation (19) for $A(\mu)$ and inserting into (20), some cancellations, and application of the Poincaré-Bertrand-formula [4] we obtain

$$\begin{aligned}h(\mu) \equiv g_2(\mu) &+ \frac{g_1(\mu\,\sigma)}{c_{12}}\sigma\,\Theta(1/\sigma-\mu)\,f_1(\mu) + \frac{c\,\sigma}{c_{12}}P\int_0^{1/\sigma}\frac{\eta\,g_1(\eta\,\sigma)}{\eta-\mu}\mathrm{d}\eta\\ &\pm\frac{c\,\sigma}{c_{12}}\int_0^{1/\sigma}\frac{\eta\,g_1(\eta\,\sigma)\,e^{-2R/\eta}}{\eta+\mu}\mathrm{d}\eta = c\int_0^1\eta\,W(\eta)\,k(\mu,\eta)\,\mathrm{d}\eta\\ &+\frac{\Omega^+(\mu)+\Omega^-(\mu)}{2}W(\mu) + \frac{\Omega^+(\mu)-\Omega^-(\mu)}{2\,\pi\,i\,\mu}P\int_0^1\frac{\eta\,W(\eta)}{\eta-\mu}\mathrm{d}\eta\end{aligned} \tag{22}$$

with
$$W(\mu) \equiv \frac{c}{c_{12}} B(\mu)\, \Theta(1/\sigma - \mu) + D(\mu)\, \Theta(\mu - 1/\sigma)$$

and
$$\begin{aligned} k(\mu,\eta) \equiv & \frac{\left\{\mu \ln \frac{\mu\sigma+1}{\mu\sigma} - \eta \ln \frac{\eta\sigma+1}{\eta\sigma}\right\}}{(\eta-\mu)} \pm \frac{\mu}{\eta+\mu} e^{-2R/\eta} \ln \left|\frac{\mu\sigma-1}{\mu\sigma}\right| \\ & \pm \frac{\eta}{\eta+\mu} e^{-2R/\eta} \ln \frac{\eta\sigma+1}{\eta\sigma} \pm \frac{e^{-2R/\eta}}{\eta+\mu} f_2(\eta) + E_1(2R\sigma)(e^{-2R/\eta} \mp 1) \\ & \pm \frac{e^{-2R/\eta}}{\eta-\mu}\left(e^{-2R/\mu} \mu E_1\left(2R\frac{\mu\sigma+1}{\mu}\right) - e^{-2R/\eta} \eta E_1\left(2R\frac{\eta\sigma+1}{\eta}\right)\right) \\ & + \frac{1}{\eta+\mu}\left(e^{-2R/\mu} \mu E_1\left(2R\frac{\mu\sigma+1}{\mu}\right) + e^{-2R/\eta} \eta E_1\left(2R\frac{\eta\sigma-1}{\eta}\right)\right). \end{aligned} \tag{23}$$

The solution of the singular integral Eq. (22) is obtained according to standard procedures, with the help of the definition
$$N(z) \equiv \frac{1}{2\pi i} \int_0^1 \frac{\eta\, W(\eta)}{\eta - z} d\eta\,, \tag{24}$$
where $N(z)$ is an analytic function in the plane cut from 0 to 1 along the real axis, with $N(z) \sim -1/z$ for $z \to \infty$.

Application of Plemelj's formulae to (24) yields
$$N^+(\mu) + N^-(\mu) = \frac{1}{\pi i} P \int \frac{\eta\, W(\eta)}{\eta - \mu} d\eta \tag{25}$$
and
$$N^+(\mu) - N^-(\mu) = \mu\, W(\mu), \tag{26}$$
from which
$$\Omega^+(\mu)\, N^+(\mu) - \Omega^-(\mu)\, N^-(\mu) = \mu\, h'(\mu) \tag{27}$$
with
$$h'(\mu) \equiv h(\mu) - c \int_0^1 \eta\, W(\eta)\, k(\mu,\eta)\, d\eta$$
and
$$\begin{aligned} \Omega^\pm(\mu) = & 1 + c_{11}\mu L(\mu\sigma) + c_{22}\mu L(\mu) \\ & + c\mu^2 L(\mu\sigma) L(\mu) - c\mu^2 \pi^2 \Theta(1/\sigma - \mu) \\ & \pm i\,(c_{11}\pi\mu\,\Theta(1/\sigma - \mu) + c_{22}\pi\mu \\ & + c\mu^2 \pi L(\mu)\, \Theta(1/\sigma - \mu) \\ & + c\mu^2\pi L(\mu\sigma)). \end{aligned} \tag{28}$$

We are therefore led to the solution of the homogeneous Hilbert problem
$$G(\mu) \equiv \Omega^+(\mu)/\Omega^-(\mu) = X^+(\mu)/X^-(\mu), \tag{29}$$
one solution of which, obeying the Hölder condition at the endpoints 0, 1 is
$$X(z) = \frac{1}{(1-z)^M} \exp\left\{\frac{1}{2\pi i} \int_0^1 \frac{\ln G(\mu)}{\mu - z} d\mu\right\}, \tag{30}$$
where $2M$ is the number of the discrete eigenvalues. From this solution we obtain
$$X^+(\mu)\, N^+(\mu) - X^-(\mu)\, N^-(\mu) = \gamma(\mu)\, h'(\mu) \tag{31}$$
with
$$\gamma(\mu) \equiv \mu\, (X^-(\mu)/\Omega^-(\mu)), \tag{32}$$
from which
$$N(z) = \frac{1}{2\pi i\, X(z)} \int_0^1 \frac{\gamma(\mu)\, h'(\mu)}{\mu - z} d\mu\,. \tag{33}$$

From the behaviour of $N(z)$ for $z \to \infty$ and the Laurent expansion
$$\int_0^1 \frac{\gamma(\mu)\, h'(\mu)}{\mu - z} d\mu = -\frac{1}{z} \int_0^1 \gamma(\mu)\, h'(\mu)\, d\mu - \frac{1}{z^2} \int_0^1 \mu\, \gamma(\mu)\, h'(\mu)\, d\mu - \ldots \tag{34}$$
we find
$$\int_0^1 \mu^i \gamma(\mu)\, h'(\mu)\, d\mu = 0 \qquad \text{for} \quad i = 0, 1, \ldots, M-1 \tag{35}$$
since
$$X(z) \sim 1/z^M \quad \text{for} \quad z \to \infty\,. \tag{36}$$
From (26) and (33) we find the following Fredholm integral equation of the second kind:
$$\begin{aligned} & W(\mu) + \frac{c}{2\,\Omega^+(\mu)\, \Omega^-(\mu)} \int_0^1 \eta\, W(\eta) \left[k(\mu,\eta)\, (\Omega^+(\mu) + \Omega^-(\mu)) - \frac{\Omega^+(\mu) - \Omega^-(\mu)}{\pi i\, \gamma(\mu)} P \int_0^1 \frac{k(\eta',\eta)\, \gamma(\eta')}{\eta' - \mu} d\eta'\right] d\eta \\ & = \frac{1}{2\,\Omega^+(\mu)\, \Omega^-(\mu)} \left((\Omega^+(\mu) + \Omega^-(\mu))\, h(\mu) - \frac{\Omega^+(\mu) - \Omega^-(\mu)}{\pi i\, \gamma(\mu)} P \int_0^1 \frac{\gamma(\eta)\, h(\eta)}{\eta - \mu} d\eta\right) \end{aligned} \tag{37}$$

with conditions following from (35)

$$\int_0^1 \mu^i \gamma(\mu)\, h(\mu)\, d\mu = c \int_0^1 \mu^i \gamma(\mu)\, d\mu \int_0^1 \eta\, W(\eta)\, k(\mu, \eta)\, d\eta \,. \tag{38}$$

Some integrations can be performed by the use of the following X-function identities:

$$X(-\mu) = \exp\left\{-\int_0^1 \frac{1}{2\pi i} \cdot \frac{d}{d\eta} (\ln G(\eta)) \ln(\eta+\mu)\, d\eta\right\}, \tag{39}$$

$$\frac{X^+(\mu)}{\Omega^+(\mu)} = \frac{1}{X(-\mu) \prod_{s=0}^{M} (\eta_s^2 - \mu^2)\, \Omega(\infty)}\,, \quad \mu > 0\,, \tag{40}$$

the proofs of which parallel closely those in the monoenergetic theory (3).

From (38) we get

$$A_1 = \frac{c \int_0^1 \gamma(\mu)\, d\mu \int_0^1 \eta\, W(\eta)\, k(\mu,\eta)\, d\eta - \int_0^1 h_0(\mu)\, \gamma(\mu)\, d\mu}{\int_0^1 h_1(\mu)\, \gamma(\mu)\, d\mu} \tag{41}$$

with

$$h_j(\mu) \equiv b_j(\mu) + \frac{a_j(\mu\,\sigma)}{c_{12}}\, \sigma\, \Theta(1/\sigma - \mu)\, f_1(\mu) + \frac{c\,\sigma}{c_{12}} P \int_0^{1/\sigma} \frac{\eta\, a_j(\eta\,\sigma)}{\eta-\mu}\, d\eta \pm \frac{c\,\sigma}{c_{12}} \int_0^{1/\sigma} \frac{\eta\, a_j(\eta\,\sigma)\, e^{-2R/\eta}}{\eta+\mu}\, d\eta \tag{42}$$

and the critical equation for the radius R

$$\int_0^1 h_0(\mu)\, \mu\, \gamma(\mu)\, d\mu + A_1 \int_0^1 h_1(\mu)\, \mu\, \gamma(\mu)\, d\mu = c \int_0^1 \mu\, \gamma(\mu)\, d\mu \int_0^1 \eta\, W(\eta)\, k(\mu,\eta)\, d\eta \quad \text{for } M=2, \tag{43}$$

where the following abbreviations have been introduced:

$$a_j(\mu) = \begin{cases} [2\, c_{12}\, \eta_j/(\mu^2 - \eta_j^2\, \sigma^2)]\, (\mu \sin(R/\eta_j) - \eta_j\, \sigma \cos(R/\eta_j)) & \text{for slabs,} \\ [2\, c_{12}\, \eta_j/(\mu^2 - \eta_j^2\, \sigma^2)]\, (\eta_j\, \sigma \sin(R/\eta_j) + \mu \cos(R/\eta_j)) & \text{for spheres.} \end{cases} \tag{44}$$

$$b_j(\mu) = \begin{cases} [2\, \eta_j/(\mu^2 - \eta_j^2)]\, (\mu \sin(R/\eta_j) - \eta_j \cos(R/\eta_j))\, (c_{22} - \eta_j \arctan[2\, \eta_j\, \sigma/(\eta_j^2\, \sigma^2 - 1)]) & \text{for slabs,} \\ [2\, \eta_j/(\mu^2 - \eta_j^2)]\, (\eta_j \sin(R/\eta_j) + \mu \cos(R/\eta_j)) & \text{for spheres,} \end{cases} \tag{45}$$

For $M=1$ we find $A_1 = 0$, and the critical equation

$$\int_0^1 h(\mu)\, \gamma(\mu)\, d\mu = c \int_0^1 \gamma(\mu)\, d\mu \int_0^1 \eta\, W(\eta)\, k(\mu,\eta)\, d\eta \,. \tag{46}$$